

NON-UPPER-SEMICONTINUITY OF ALGEBRAIC DIMENSION FOR FAMILIES OF COMPACT COMPLEX MANIFOLDS

A. FUJIKI AND M. PONTECORVO

ABSTRACT. In this note we show that in a certain subfamily of the Kuranishi family of any half Inoue surface the algebraic dimensions of the fibers jump downwards at special points of the parameter space showing that the upper semi-continuity of algebraic dimensions in any sense does not hold in general for families of compact non-Kähler manifolds. In the Kähler case, the upper semi-continuity always holds true in a certain weak sense.

1. STATEMENT OF RESULTS

In this note we shall show the following theorem which gives examples of holomorphic families of compact complex surfaces whose algebraic dimensions jump downwards under specializations.

Theorem 1.1. *Let S be a half Inoue surface with second Betti number m and C the unique twisted anti-canonical curve on it. Let $g : (S, C) \rightarrow T$, $(S_o, C_o) = (S, C)$, $o \in T$, be the Kuranishi family of deformations of the pair (S, C) . Then the Kuranishi space T is smooth of dimension m and contains a divisor with normal crossings $A = \bigcup_{i=1}^m A_i$ with m smooth irreducible components passing through the base point o such that the following hold: the fiber $S_t, t \neq o$, is a blown-up half Inoue surface if $t \in A$, and is a blown-up elliptic diagonal Hopf surface if $t \notin A$.*

We refer to Proposition 1.3 below for more precise structure of the surface S_t for $t \notin A$. Recall that the *algebraic dimension* of a compact connected complex surface is the transcendence degree of its meromorphic function field. It takes one of the values 0, 1 and 2. Any half Inoue surface has algebraic dimension zero, while any elliptic diagonal Hopf

¹Research partially supported by Grant-in-Aid for Scientific Research (No.18340017) MEXT, and PRIN, Italy: Metriche riemanniane e varietà differenziabili

surface is of algebraic dimension one. This is the property of our main interest in this note:

Corollary 1.2. *Let $a_t := a(S_t)$ be the algebraic dimension of S_t . Then $a_t = 0$ for $t \in A$ and $= 1$ for $t \notin A$. Thus the algebraic dimension jumps downwards at special points; in particular it is not upper semicontinuous in the parameter t .*

After some preliminaries in Section 2, we prove Theorem 1.1 in Section 3 together with the following supplement to it, which reveals special features of our examples.

Proposition 1.3. *Let the notations be as in Theorem 1.1. Then for any $t \notin A$, S_t is the blowing-up of an elliptic diagonal Hopf surface \bar{S}_t . The elliptic fibration of \bar{S}_t is smooth except for two multiple fibers with multiplicity two, and the image \bar{C}_t of C_t in \bar{S}_t is one of its smooth fibers. Moreover, S_t is obtained from \bar{S}_t by blowing up m points on \bar{C}_t .*

Example. When $m = 1$, the half Inoue surface S is unique up to isomorphisms. It contains a unique curve C , which is a rational curve with a single node. We get the Kuranishi family of deformations $\{(S_t, C_t)\}_{t \in D}$ of (S, C) parametrized by a one dimensional disc $D = \{|t| < \varepsilon\}, \varepsilon > 0$. For any $t \neq 0$, S_t is the blowing-up at one point p_t of an elliptic diagonal Hopf surface \bar{S}_t over the complex projective line \mathbf{P} . Let \bar{C}_t be the fiber of $\bar{h}_t : \bar{S}_t \rightarrow \mathbf{P}$ containing p_t . Then \bar{h}_t is smooth except for two multiple fibers with multiplicity two and C_t is the proper transform of \bar{C}_t in S_t . In particular $a_t = 1$ for $t \neq 0$ and $a_0 = 0$ (cf. Proposition 1.3 below).

Remark 1.1. For each integer $n > 1$ by taking the product of n copies S_t^n of each fiber of f in Theorem 1.1 or by considering the Douady space (Hilbert scheme) $S_t^{[n]}$ of 0-dimensional analytic subspaces of S_t of length n we get a family of $2n$ -dimensional compact complex manifolds X_t with the same parameter space T such that $a(X_t) = 0$ for $t \in A$, but $a(X_t) = n$ for $t \notin A$.

In the above results the surfaces S_t are all non-Kähler surfaces. In fact the phenomena as above never occur in a family of compact Kähler manifolds. In Section 4, for the purpose

of comparison, we give a general property of the variation of algebraic dimensions of the fibers in a family of compact Kähler manifolds.

2. PRELIMINARIES

A *cycle of rational curves* on a smooth surface is a compact connected curve C which is either an irreducible rational curve with a single node or is a reducible curve with k nodes whose irreducible components are nonsingular rational curves $C_i, 1 \leq i \leq k, k \geq 2$, such that C_i and C_{i+1} intersects at a single point and there exists no other intersections, where $C_{k+1} = C_1$ by convention.

A compact connected complex surface S is called of *class VII* if $b_1(S) = 1$ and $\kappa(S) = -\infty$, where b_1 and κ are the first Betti number and the Kodaira dimension of S respectively. Such a surface is non-Kähler and its algebraic dimension is less than two. S is called of *class VII₀⁺* if S is of class VII, is minimal, i.e., contains no (-1) -curves, and with positive second Betti number. (A (-1) -curve is a nonsingular rational curve with self-intersection number -1 .)

Inoue [7][8] constructed the first examples of surfaces of class VII₀⁺. These surfaces are called hyperbolic, parabolic and half Inoue surfaces according to Nakamura [12]. By [12, (8.1)] a surface S of class VII₀⁺ is a hyperbolic (resp. parabolic) Inoue surface if and only if S contains two cycles of rational curves C_1 and C_2 (resp. a cycle of rational curves C_0 and a smooth elliptic curve E). Similarly, S is a half Inoue surface if and only if S contains a cycle C of rational curves and there exists an unramified double covering $u : \tilde{S} \rightarrow S$ such that $u^{-1}(C)$ is a disjoint union of two cycles of rational curves \tilde{C}_1 and \tilde{C}_2 which are mapped isomorphically onto C (cf. [12, (1.6)(9.2.4)]). \tilde{S} is then a hyperbolic Inoue surface. All these Inoue surfaces have infinite cyclic fundamental group. By [8] the algebraic dimension of these surfaces equals zero.

In the hyperbolic (resp. parabolic) case denote by C the union of C_1 and C_2 (resp. C_0 and E). Then for any Inoue surface there exists no irreducible curves other than the irreducible components of C . Moreover, in the hyperbolic or parabolic Inoue case C is the unique *anti-canonical curve* on S , i.e., C is the unique member of the anti-canonical system $|-K_S|$. Similalry, in the half Inoue case C is the unique *L-twisted anti-canonical*

curve in the sense that C is the unique member of the L -twisted anti-canonical system $|-(K_S + L)|$ for a unique non-trivial holomorphic line bundle L with $L^{\otimes 2}$ trivial.

A *diagonal Hopf surface* is a Hopf surface $S = S(\alpha, \beta)$ obtained as the quotient of $\mathbf{C}^2 - 0$ by an infinite cyclic group generated by the diagonal transformation $(z, w) \rightarrow (\alpha z, \beta w)$ for complex numbers α, β with $0 < |\alpha|, |\beta| < 1$. The images of z - and w - axes give two canonical elliptic curves E_1 and E_2 on S and we have $E_1 + E_2 = -K_S$ (cf. [10, (97)]). If S admits a non-constant meromorphic function, it admits a unique structure of an elliptic surface $h : S \rightarrow \mathbf{P}$ over the projective line \mathbf{P} and any irreducible curve on S is smooth and is a fiber of h up to multiplicity (cf. [10]). Recall also that S has the vanishing second Betti number.

We also quote two lemmas [5, Lemmas 3.2, 3.3] for later purpose for the convenience of the reader. The first one is originally due to Nakamura and the second one easily follows from the first one.

Lemma 2.1. *Let \hat{S} be a compact complex surface of class VII_0 with infinite cyclic fundamental group. Suppose that \hat{S} admits a disconnected anti-canonical curve. Then \hat{S} is either a hyperbolic or parabolic Inoue surface or a diagonal Hopf surface.*

Lemma 2.2. *Let \tilde{S} be a compact complex surface of class VII with infinite cyclic fundamental group. Suppose that \tilde{S} admits a disconnected anti-canonical curve \tilde{C} . Then the minimal model \hat{S} of \tilde{S} is either a hyperbolic or parabolic Inoue surface or a diagonal Hopf surface, and $\tilde{S} \rightarrow \hat{S}$ is obtained by blowing up \hat{S} at a finite number of points (possibly infinitely near) on the image \hat{C} of \tilde{C} . Moreover, \hat{C} is an anti-canonical curve on \hat{S} .*

Remark 2.1. It is known that the fundamental group of a known surface $S \in VII_0$ is always infinite cyclic unless S is a Hopf surface; in the latter case however S is always finitely covered by a primary Hopf surface, that is one with $\pi_1(S) = \mathbf{Z}$.

3. PROOF OF MAIN RESULTS

We consider the Kuranishi family $g : (\mathcal{S}, \mathcal{C}) \rightarrow T$, $(S_o, C_o) = (S, C)$, $o \in T$ of deformations of the pair (S, C) as in Theorem 1.1. This was studied in our previous paper

[5], to which we refer for the details. Indeed, except the assertion that $S_t, t \notin A$, has the structure of an elliptic surfaces with special structures, all statements in the theorem are given in Proposition 3.14 of [5] partly without proof, as an analogue of the corresponding result in the hyperbolic case ([5, Proposition 3.13]).

We first prove Lemma 3.1 below which is a detailed version of Lemma 3.4 of [5] also stated without proof there. Let S in general be a compact complex surface of class VII with infinite cyclic fundamental group. Then there exist a unique unramified double covering $u : \tilde{S} \rightarrow S$ and a unique non-trivial holomorphic line bundle $L = L_S$ with $L^{\otimes 2}$ trivial such that u^*L is trivial. In fact they are both associated to the representation of the fundamental group $\pi_1(S) \cong \mathbf{Z} \rightarrow \mathbf{Z}_2 = \{\pm 1\} \subseteq \mathbf{C}^*$.

Lemma 3.1. *Let the notations be as above. Suppose that S contains an L -twisted connected anti-canonical curve C such that $\tilde{C} := u^{-1}(C)$ is disconnected in \tilde{S} . Then the minimal model \bar{S} of S is either a half Inoue surface or an elliptic diagonal Hopf surface. In the latter case the elliptic fibration of \bar{S} is smooth except for two multiple fibers with multiplicity two, and the image \bar{C} of C in \bar{S} is one of the smooth fibers. S is obtained by blowing up points on \bar{C} (unless $S = \bar{S}$) and C is the proper transform of \bar{C} .*

Proof. Let \hat{S} be the minimal model of \tilde{S} . Let $\tilde{v} : \tilde{S} \rightarrow \hat{S}$ and $v : S \rightarrow \bar{S}$ be the blowing-down maps. For any (-1) -curve B on S , $u^{-1}(B)$ is a disjoint union of two (-1) -curves \tilde{B}_1 and \tilde{B}_2 on \tilde{S} . Thus contracting successively all the (-1) -curves on \tilde{S} obtained in this way we get a blowing-down map $\tilde{v}' : \tilde{S} \rightarrow \hat{S}'$. We show that \hat{S}' coincides with \hat{S} . Let ι be the Galois involution for u . Suppose that there exists a (-1) -curve \tilde{B} on \hat{S}' . $\iota(\tilde{B})$ is a (-1) -curve, but by the definition of \hat{S}' it must intersect with \tilde{B} so that $D := \tilde{B} \cup \iota(\tilde{B})$ is connected with intersection number $\tilde{B} \cdot \iota(\tilde{B}) \geq 2$. Then $D^2 \geq 2$, which is impossible since $a(\hat{S}') = a(S) \leq 1$ (cf. [9, Th.8]). Hence $\hat{S}' = \hat{S}$ and we get the induced double covering $\hat{u} : \hat{S} \rightarrow \bar{S}$ with Galois involution denoted by $\hat{\iota}$.

On the other hand, since $C = -(K_S + L)$ by our assumption, we have $\tilde{C} = -K_{\tilde{S}}$. \tilde{C} is thus an anti-canonical curve on \tilde{S} , and hence by Lemma 2.2 the image \hat{C} of \tilde{C} in \hat{S} under $\tilde{v} : \tilde{S} \rightarrow \hat{S}$ is again a disconnected anti-canonical curve on \hat{S} and \tilde{v} is obtained by blowing-up points on \hat{C} . Moreover, each of the two connected components $\hat{C}_\alpha, \alpha = 1, 2$,

of \hat{C} is mapped isomorphically onto the image \bar{C} on \bar{S} . Thus by Lemma 2.1 \hat{S} is either a hyperbolic Inoue surface or a diagonal Hopf surface and we have $-K_{\hat{S}} = \hat{C}_1 + \hat{C}_2$. (Since $\hat{C}_1 \cong \hat{C}_2$, \hat{S} is never a parabolic Inoue surface.) Therefore in the first case \bar{S} is a half Inoue surface, and in the second case it is a Hopf surface with infinite cyclic fundamental group, i.e., a primary Hopf surface.

We now consider the second case in more detail. There are two types of primary Hopf surfaces as exhibited in [10, (94)(95)], of which one type consists precisely of diagonal Hopf surfaces. These two types are preserved under a finite unramified covering, as follows easily from their defining formulae (loc.cit). Thus \bar{S} is a diagonal Hopf surface as well as \hat{S} , and the first assertion is proved. Note also that if $\bar{S} = S(\alpha, \beta)$, then \hat{S} is identified with $S(\alpha^2, \beta^2)$.

We further show that \bar{S} is an elliptic surface. Note first that each \hat{C}_α is a smooth elliptic curve as well as \bar{C} . On the other hand, since \bar{S} is a diagonal Hopf surface, \bar{S} admits two elliptic curves \bar{E}_1 and \bar{E}_2 such that $-K_{\bar{S}} = \bar{E}_1 + \bar{E}_2$. Moreover, if $\hat{E}_i := \hat{u}^{-1}(\bar{E}_i)$, $i = 1, 2$, these two curves are again the canonical elliptic curves on the diagonal Hopf surface \hat{S} , and hence we have $\hat{E}_1 + \hat{E}_2 = -K_{\hat{S}} = \hat{C}_1 + \hat{C}_2$. Note that $\hat{E}_1 + \hat{E}_2 \neq \hat{C}_1 + \hat{C}_2$ since their images $\bar{E}_1 + \bar{E}_2$ and \bar{C} in \bar{S} are different. Thus the anti-canonical system $| -K_{\hat{S}} |$ has positive dimension and so the algebraic dimension of \hat{S} is positive. Hence \hat{S} admits a unique elliptic fibration $\hat{h} : \hat{S} \rightarrow \mathbf{P}$ such that the above curves \hat{E}_i , $i = 1, 2$, and \hat{C}_α , $\alpha = 1, 2$, become fibers of \hat{h} .

On the other hand, since the arithmetic genus $\chi(O_{\hat{S}})$ of \hat{S} vanishes, from the canonical bundle formula for elliptic surfaces [9, Th.12], we deduce easily

$$(1) \quad -\hat{K} = \hat{h}^*O(2) - \sum_{\nu} (m_{\nu} - 1)F_{\nu},$$

where $O(l)$ is the line bundle of degree l on \mathbf{P} , and F_{ν} , $0 \leq \nu \leq b$, are multiple fibers with multiplicity $m_{\nu} > 1$. (In the case of a diagonal Hopf surface we have $0 \leq b \leq 2$ by [10, Th.31].) Thus, if $b = 2$, noting that $m_{\nu}F_{\nu} = \hat{h}^*O(1)$ we have $-\hat{K} = F_1 + F_2$. Since F_i do not move in \hat{S} , this implies that $\dim | -\hat{K} | = 0$, contradicting what we have obtained. Thus $b \neq 2$.

Next we show that $b \neq 1$. In fact, suppose that $b = 1$ so that $-\hat{K} = \hat{h}^*O(1) + F_1$. Then neither of \hat{C}_α is a multiple fiber since the multiple fiber is fixed by \hat{i} while $\hat{i}(\hat{C}_1) = \hat{C}_2$. Thus $\hat{C}_\alpha = \hat{h}^*O(1)$ and hence $-\hat{K} = \hat{C}_1 + \hat{C}_2 = \hat{h}^*O(2)$, which is a contradiction to (1) since $b = 1$. Thus $b = 0$ and \hat{h} is a principal elliptic bundle.

Now $\bar{\iota}$ induces a non-trivial involution on \mathbf{P} making \hat{h} equivariant, and we have the induced morphism $\bar{h} : \bar{S} \cong \hat{S}/\langle \iota \rangle \rightarrow \mathbf{P}/\langle \iota \rangle \cong \mathbf{P}$ giving the elliptic fibering structure on \bar{S} . Thus \bar{h} has exactly two multiple fibers of multiplicity two over the two fixed points of $\bar{\iota}$ on \mathbf{P} and is otherwise smooth.

Finally, each of the inverse images in \hat{S} of the two multiple fibers of \bar{h} is \hat{i} -invariant, while we have $\hat{i}(\hat{C}_1) = \hat{C}_2$. Thus \bar{C} is a smooth fiber of \bar{h} . The last assertion immediately follows from the corresponding assertion for \bar{v} and \bar{C} proved above. \square

Proof of Theorem 1.1 and Proposition 1.3. That T is smooth of dimension m was shown in Proposition 3.12 of [5]. The hypersurfaces $A_i, 1 \leq i \leq m$, of the theorem are $T(p)$ in [5, Prop.3.12] defined for each node p of C . In fact C is a cycle of rational curves with m irreducible components. So it admits precisely m nodes. $T(p)$ is the locus of the point $t \in T$ such that the node p remains to be a node in C_t . (That is, along $T(p)$ the deformation of the isolated singularity germ (C, p) is trivial.) Thus C_t is still a cycle of rational curves if $t \in A$ and is a nonsingular elliptic curve if $t \notin A$.

Now we take the unique unramified double covering $u : \tilde{S} \rightarrow S$ over S and let $\tilde{C} := u^{-1}(C)$. \tilde{C} has two connected components each of which is mapped isomorphically onto C . We can extend this covering to a relative double covering of the family so that for each t we have the unramified double covering $u_t : \tilde{S}_t \rightarrow S_t$. Then similarly $\tilde{C}_t := u_t^{-1}(C_t)$ has two connected components, each mapped isomorphically onto C_t . Moreover, by Proposition 3.13 of [5] \tilde{C}_t is the anti-canonical curve on \tilde{S}_t . Let L_t be the unique non-trivial holomorphic line bundle on S_t with $L_t^{\otimes 2}$ trivial, i.e., $L_t = L_{S_t}$, which depends holomorphically on t with $L_o = L$.

C_t is then an L_t -twisted anti-canonical curve on S_t . Indeed, $u_t^*(K_t + C_t) = \tilde{K}_t + \tilde{C}_t = 0$ on \tilde{S}_t . Thus we get $K_t + C_t = L_t$ or 0 . But since $L_o = L$, by continuity the former holds true for any t as desired. Then by Lemma 3.1 the minimal model \bar{S}_t of S_t is either a half

Inoue surface or a diagonal elliptic Hopf surface. By the above description of the curve C_t these two cases occur precisely when $t \in A$ and $t \notin A$ respectively. Together with Lemma 3.1 the theorem and the proposition follow. (If $t \notin A$, $b_2(\bar{S}_t) = 0$ and $b_2(S_t) = m$. Thus the blowing-up occurs at m points on $\bar{C}_t \subseteq \bar{S}_t$.) \square

Remark 3.1. Lemma 3.4 and Proposition 3.14 in [5] follow clearly from Proposition 1.3 and Lemma 3.1 above. In fact, the arguments above are completely the same even if we start from a properly blown-up half Inoue surface instead of just a half Inoue surface as we did in [5].

Remark 3.2. 1) The above results can also be stated in terms of the Kuranishi family

$$(2) \quad \tilde{g} : (\tilde{S}, \tilde{C}) \rightarrow \tilde{T}, \quad (\tilde{S}_o, \tilde{C}_o) = (\tilde{S}, \tilde{C}), \quad o \in \tilde{T},$$

of deformations of the pair (\tilde{S}, \tilde{C}) , where, as in the above proof, $u : \tilde{S} \rightarrow S$ is the canonical unramified double covering of S with the Galois involution ι on \tilde{S} and $\tilde{C} = u^{-1}(C)$. \tilde{T} is known to be smooth of dimension $2m$ and the family is universal ([5, Prop.3.13]). Thus ι extends to an involution $\tilde{\iota}$ on the family \tilde{g} . Let \tilde{T}^ι be the fixed point set of \tilde{T} . Then the restriction of this family to \tilde{T}^ι gives the universal family of deformations of the triple $(\tilde{S}, \tilde{C}, \iota)$, and the (fiberwise) quotient by $\tilde{\iota}$ of the restricted family to \tilde{T}^ι is naturally identified with the Kuranishi family g of the pair (S, C) considered above. Thus the same statement as Corollary 1.2 holds true also for the family over \tilde{T}^ι above.

It may be interesting to ask if such a phenomenon is accidental or a rather general one. For instance we may ask the following question: Starting from any hyperbolic Inoue surface \tilde{S} (which is in general not an unramified double covering of a half Inoue surface), consider the Kuranishi family (2) of deformations of (\tilde{S}, \tilde{C}) , where \tilde{C} is the unique anti-canonical curve on \tilde{S} . Then can we find a subspace \tilde{T}' of \tilde{T} with $o \in \tilde{T}'$ such that when the family is restricted to \tilde{T}' we get the non-upper-semicontinuity of algebraic dimension as in Corollary 1.2 ? How can we characterize geometrically such families ?

2) Some holomorphic line bundles defined on $g^{-1}(T - A)$ do not extend to the total space \mathcal{S} of the Kuranishi family of Theorem 1.1. Let us discuss this phenomenon in the simplest case $m = 1$ (cf. Example in Section 1). In this case for $t \neq o$, S_t has a canonical

structure of an elliptic surface $f_t : S_t \rightarrow \mathbf{P}$. We consider the line bundle $H_t := f_t^*O(1)$ on S_t . C_t is an irreducible component of a fiber, say F_b , of f_t over the point $b = b_t \in \mathbf{P}$; in fact $F_b = C_t + E_t$ for a unique (-1) -curve E_t on S_t . Obviously, K_t, L_t and $C_t, t \neq o$, extend holomorphically to $S = S_o$, while F_b and E_t do not. More precisely we have

Claim. The line bundles F_t and $[E]_t, t \neq o$, never extend holomorphically to a line bundle on $S = S_o$, where $[E]_t$ is the line bundle defined by E_t .

Proof. Suppose that F_t extends holomorphically to F_o on S . Then by the upper semicontinuity we have $2 = h^0(F_t) \leq h^0(F_o)$, where $h^0(F_s) = \dim H^0(F_s), s \in D$. Since $a(S) = 0$, we must have $h^0(F_o) \leq 1$, which is a contradiction. The result for $[E]_t$ then follows from the relation $F_b = C_t + E_t$. Or similarly to the above, the extension implies that $1 = h^0([E]_t) \leq h^0([E]_o) \leq 1$. Then a nonzero section of $h^0([E]_o)$ would give a curve other than C on S , which is a contradiction. \square

It seems interesting to study the behaviour of E_t and $[E]_t$ when t tends to o .

4. KÄHLER CASE

The phenomenon as in Theorem 1.1 never occurs in the Kähler category. We shall give in this section a proof of this fact known certainly to experts for the convenience of the reader. Namely we prove:

Proposition 4.1. *Let $f : X \rightarrow T$ be a smooth holomorphic family of compact connected Kähler manifolds parametrized by a connected complex manifold T . Suppose that T is simply connected or $\dim T = 1$. For any nonnegative integer k let T_k be the subset of T consisting of points $t \in T$ such that the fiber $X_t := f^{-1}(t)$ is of algebraic dimension $a(X_t) \geq k$. Then T_k is the union of at most a countable number of analytic subsets of T .*

In general the sets $T_k, k > 0$, are not closed so that algebraic dimension is not upper semi-continuous with respect to the classical topology even in the Kähler case. T_k can actually be dense as the following examples show.

Example. 1) Let S be a K3 surface or a complex torus of dimension two. Let $f : \mathcal{S} \rightarrow T, o \in T, S_o = S$, be the Kuranishi family of S , where T is a smooth germ of dimension

20 and 4 respectively. Then the locus $T^+ := T_1 \cup T_2$ in T where the fiber is of positive algebraic dimension is a countable union of smooth hypersurfaces. Moreover, T^+ is dense in T .

2) Let S be as above. For each Kähler class on S we have the associated twistor space Z which is a smooth fiber space over the complex projective line \mathbf{P}^1 . The locus B^+ of \mathbf{P} where the fiber is of positive algebraic dimension is a countable dense subset of \mathbf{P} . For an explicit example of the set T^+ see e.g. [4, Prop.5.7].

In 1) and 2) above the points corresponding to projective surfaces are even dense in the base space. This is a consequence of [3, Th.4.8], which in general holds for any compact hyperkähler manifolds. Similar phenomena are also common in the non-Kähler case. For instance the diagonal Hopf surfaces are parametrized by $(\alpha, \beta) \in D^* \times D^* =: T$ with the corresponding surface $S(\alpha, \beta)$ given by $(\mathbf{C}^2 - \{0\})/\langle(\alpha, \beta)\rangle$, where D^* is the punctured unit disc. By [10, Th.31] $a(S(\alpha, \beta)) = 1$ if and only if $\alpha^m = \beta^n$ for some $(m, n) \in \mathbf{Z}^2$. T^+ then forms a dense countable family of analytic subsets of $D^* \times D^*$.

Now for the proof of Proposition 4.1 we first note that for any compact connected complex manifold X the algebraic dimension $a(X)$ is described as

$$(3) \quad a(X) = \max\{\kappa(L); L \in \text{Pic}X\}$$

where $\kappa(L)$ is the Iitaka (or L -) dimension of the line bundle L and $\text{Pic}X$ is the Picard variety of X (cf. [13]). Next, we recall the following result of Lieberman and Sernesi [11].

Lemma 4.2. *Let $f : Y \rightarrow S$ be a proper smooth morphism of irreducible complex spaces with connected fibers. Let L be a holomorphic line bundle on Y . For any non-negative integer k we set $S_k(L) := \{s \in S; \kappa(L_s) \geq k\}$, where L_s is the restriction of L to the fiber Y_s . Let $k_0 := \min\{k; S_k(L) \neq S\}$. Then $S_{k_0}(L)$ is the union of at most a countable number of analytic subvarieties of S .*

By applying the above lemma to each of the subvarieties of which $S_{k_0}(L)$ is a union, we get that $S_{k_0+1}(L)$ again is the union of at most a countable number of proper analytic subvarieties of S . Proceeding in the same way, we obtain the following:

Corollary 4.3. *For each $k \geq 0$, $S_k(L)$ is the union of at most countably many analytic subvarieties of S .*

Proof of Proposition 4.1. Let $f : X \rightarrow T$ be as in the proposition. Suppose first that T is simply connected. Then the local system $R^2 f_* \mathbf{Z}$ is trivial and is identified with the trivial system $T \times \Gamma \rightarrow T$, where $\Gamma := H^2(X_o, \mathbf{Z})$ for some fixed reference point $o \in T$. Any element γ of Γ thus defines a (constant) section $s(\gamma)$ of $R^2 f_* \mathbf{Z}$. We consider the long exact sequence

$$\rightarrow R^1 f_* O_X^* \rightarrow R^2 f_* \mathbf{Z} \rightarrow R^2 f_* O_X \rightarrow$$

obtained by applying Rf_* to the exponential sequence on X

$$0 \rightarrow \mathbf{Z} \rightarrow O_X \rightarrow O_X^* \rightarrow 0.$$

For any $\gamma \in \Gamma$, $s(\gamma)$ is mapped to a section $b(\gamma)$ of $R^2 f_* O_X$. Since X_t are all Kähler, $R^2 f_* O_X$ is a locally free O_T -module and hence the zero locus of $b(\gamma)$ is an analytic subset T_γ of T . Also by the Kähler assumption, $h^i(O_{X_t})$ is constant for all i and therefore O_X is cohomologically flat in all dimensions with respect to f [1, Th.4.1.2]. In particular T_γ coincides with the set of points $t \in T$ such that the restriction $s(\gamma)_t \in H^2(X_t, \mathbf{Z})$ is of type $(1, 1)$ on X_t , or equivalently, $s(\gamma)_t$ is the first chern class of a holomorphic line bundle on X_t .

Let $p : P = \text{Pic}X/T \rightarrow T$ be the relative Picard variety associated to the morphism f (cf. the remarks at the end of [6, §3] and [2]). Let $p_\gamma : P_\gamma \rightarrow T$ be the component of P parametrizing line bundles with chern class γ . Then the image of p_γ is precisely the set T_γ , and p_γ is proper, smooth and with connected fibers over T_γ again by the Kähler condition. (Here we consider T_γ as a reduced complex space.) Take the fibered product $g_\gamma : X_\gamma := X \times_T P_\gamma \rightarrow P_\gamma$ with respect to p_γ . First assume that f admits a holomorphic section. Then such a section gives rise to the tautological line bundle L_γ on X_γ such that $(L_\gamma)_y, y \in P_\gamma$, is precisely the holomorphic line bundle on $X_{\gamma,y} \cong X_{p_\gamma(y)}$ corresponding to y (cf. [6]).

Then by Corollary 4.3 applied to (g_γ, L_γ) , for any $k \geq 0$ the set $S_k(L_\gamma)$ is the union of at most countably many analytic subvarieties of P_γ . Then since p_γ is proper, by Remmert

$\bar{S}_k(L_\gamma) := p_\gamma(S_k(L_\gamma))$ also is the union of at most countably many analytic subvarieties of T , and hence so is the union

$$A_k := \bigcup_{\gamma \in \Gamma} \bar{S}_k(L_\gamma) = \{t \in T; \kappa(L_t) \geq k \text{ for some } L_t \in \text{Pic } X_t\}.$$

On the other hand, by (3) we have $a(X_s) \geq k$ if and only if $s \in A_k$. Namely, $A_k = T_k$ and the proposition is proved in this case.

When f does not admit a holomorphic section, we consider the pull-back f_X of f to X via $f : X \rightarrow T$, namely f_X is the projection $f_X : X \times_T X \rightarrow X$ to the second factor. Since f_X admits a canonical holomorphic section, we can apply what we have proved above to f_X and get that $X_k := \{x \in X; a(X_x) \geq k\}$ is at most a countable union of analytic subvarieties of X for all k . In view of the relation $f^{-1}(T_k) = X_k$ coming from the definitions of T_k and X_k , we deduce the same conclusion for T_k . Thus the proposition is proved when T is simply connected.

In the general case, we consider the pull-back $f_{\tilde{T}} : X \times_T \tilde{T} \rightarrow \tilde{T}$ of f to the universal covering \tilde{T} of T . By what we have proved above, \tilde{T}_k is at most a countable union of analytic subvarieties of \tilde{T} and T_k coincides with the image of \tilde{T}_k . Now if $\dim T = 1$, \tilde{T}_k consists of at most countably many points and hence so does T_k . Thus the proposition is true also in this case.

Remark 4.1. 1) If T is not simply connected and $\dim T > 1$, the above argument fails since the image in T of an analytic subvariety of \tilde{T} is not in general analytic. We do not know if the proposition still holds for a general T .

2) The above arguments cannot be applied to the families in Theorem 1.1 since for them the components P_γ of $\text{Pic } X/T$ are not proper over the base T . Indeed, we have $P_\gamma \cong T_\gamma \times \mathbf{C}^*$.

REFERENCES

- [1] Bănică, C., and Stănăşilă, O., Algebraic methods in the global theory of complex spaces, Editura Academiei, Bucuresti, and John Wiley & Sons, London-New York-Sieney-Toronto, 1976
- [2] Bingener, J., Darstellbarkeitskriterien für analytische funktoren, Ann. Sci. École Norm. Sup., 4e série, **13** (1980), 317–347.

- [3] Fujiki, A., On primitively symplectic compact Kähler V-manifolds of dimension four, In: Classification of algebraic and analytic manifolds, (Ed. K. Ueno), Progress in Math., **39** (1983), 71-250.
- [4] Fujiki, A., Finite automorphism groups of complex tori of dimension two, Publ. RIMS, Kyoto Univ., **24** (1988), 1-97.
- [5] Fujiki, A., and Pontecorvo, M., Anti-self-dual bihermitian structures on Inoue surfaces, arXiv:math.DG/0903.1320v1, (2009).
- [6] Grothendieck, A., Techniques de construction en géométrie analytique, IX., Quelques problèmes de modules, Séminaire Henri Cartan, t.13, 1960/61, n° 16.
- [7] Inoue, M., New surfaces with no meromorphic functions, Proc. of ICM (Vancouver, B. C., 1974), 423-426.
- [8] Inoue, M., New surfaces with no meromorphic functions, II. In: Complex analysis and algebraic geometry (eds. Baily, W.L.Jr. and Shioda, T.), Iwanami Shoten Publ. and Cambridge Univ. Press, 91-106, 1977.
- [9] Kodaira, K., On the structure of compact complex analytic surfaces, I, Amer. J. Math., **86** (1964), 751-798.
- [10] Kodaira, K., On the structure of compact complex analytic surfaces, II, Amer. J. Math., **88** (1966), 682-721.
- [11] Lieberman, D., and Sernesi, E., Semicontinuity of L -dimension, Math. Ann., **225** (1977), 77-88.
- [12] Nakamura, I., On surfaces of class VII₀ with curves, Invent. Math. **78**, (1984), 393-443.
- [13] Ueno, K., Classification theory of algebraic varieties and compact complex spaces, Lecture Notes in Math., **439**, 1975.

Department of Mathematics
 Graduate School of Science
 Osaka University

Toyonaka 560-0043 Japan

E-mail address: fujiki@math.sci.osaka-u.ac.jp

Dipartimento di Matematica
 Università Roma Tre

00146 Roma, Italy

E-mail address: max@mat.uniroma3.it